Chaos and Maps in Relativistic Dynamical Systems

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Abstract: The basic work of Zaslavskii *et al* showed that the classical non-relativistic electromagnetically kicked oscillator can be cast into the form of an iterative map on the phase space; the resulting evolution contains a stochastic flow to unbounded energy. Subsequent studies have formulated the problem in terms of a relativistic charged particle in interaction with the electromagnetic field. We review the structure of the covariant Lorentz force used to study this problem. We show that the Lorentz force equation can be derived as well from the manifestly covariant mechanics of Stueckelberg in the presence of a standard Maxwell field, establishing a connection between these equations and mass shell constraints. We argue that these relativistic generalizations of the problem are intrinsically inaccurate due to an inconsistency in the structure of the relativistic Lorentz force, and show that a reformulation of the relativistic problem, permitting variations (classically) in both the particle mass and the effective "mass" of the interacting electromagnetic field, provides a consistent system of classical equations for describing such processes.

1. INTRODUCTION

Zaslavskii *et al* [1] have studied the behavior of particles in the field of a wave packet of an electric field in the presence of a static magnetic field. For a broad wave packet with sufficiently uniform spectrum, the problem can be stated in terms of an electrically kicked harmonic oscillator. They find that for rational ratios between the frequency of the kicking field and the Larmor frequency associated with the magnetic field, the phase space of the system is covered by a mesh of finite thickness; inside the filaments of the mesh, the dynamics of the particle is stochastic and outside (in the cells of stability), the dynamics is regular. This structure is called a stochastic web. It was found that this pattern covers the entire phase plane, permitting the particle to diffuse arbitrarily far into the region of high energies (a process analogous to Arnol'd diffusion [2]).

Since the stochastic web leads to unbounded energies, several authors have considered the corresponding relativistic problem. Longcope and Sudan [3] studied this system (in effectively $1 + 1/2$ dimensions) and found that for initial conditions close to the origin of the phase space there is a stochastic web, which is bounded in energy, of a form quite similar, in the neighborhood of the origin, to the non-relativistic case treated by Zaslavskii *et al* [1]. Karimabadi and Angelopoulos [4] studied the case of an obliquely propagating wave, and showed that under certain conditions, particles can be accelerated to unlimited energy through an Arnol'd diffusion in two dimensions.

The equations used by Longcope and Sudan[3] and Karimabadi and Angelopoulos[4] are derived from the well-known covariant Lorentz force

$$f^\mu = m\frac{d^2x^\mu}{ds^2} = \frac{e}{c}F^\mu{}_\nu\frac{dx^\nu}{ds}, \tag{1.1}$$

where ds is usually taken to be the "proper time" of the particle. Multiplying both side by dx_μ/ds and summing over μ (we use the Einstein summation convention that adjacent indices are summed unless otherwise indicated, and the metric is taken to be $(-, +, +, +)$ for the indices $(0, 1, 2, 3)$, distinguishing upper and lower indices), one obtains

$$\frac{dx_\mu}{ds}\frac{d^2x^\mu}{ds^2} = \frac{1}{2}\frac{d}{ds}\left(\frac{dx_\mu}{ds}\frac{dx^\mu}{ds}\right) = 0; \tag{1.2}$$

taking the usual value for the constant (in s), we have that

$$\frac{dx_\mu}{ds}\frac{dx^\mu}{ds} = -c^2. \tag{1.3}$$

This result provides a consistent identification of the parameter s on the particle trajectory (world-line) as the "proper time":

$$\begin{aligned} ds^2 &= -\frac{1}{c^2}dx_\mu dx^\mu \\ &= dt^2 - \frac{1}{c^2}d\mathbf{x}^2 \\ &= dt^2\left(1 - \frac{v^2}{c^2}\right) \end{aligned} \tag{1.4}$$

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so that

$$dt = \frac{ds}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma ds, \tag{1.5}$$

the Lorentz transformation of the time interval for a particle at rest to the interval observed in a moving frame. This formula has been used almost universally in calculations of the dynamics of relativistic charged particles [6,7]. The Lorentz trtansformation, however, applies only to inertial frames. Phenomena occurring in two inertial frames in relative motion are, according to the theory of special relativity, related by a Lorentz transformation. An accelerating frame, as pointed out by Landau and Lifshitz[6], induces a more complicated form of metric than the flat space $(-,+,+,+)$. Mashoon [11] has emphasized that the use of a sequence of instantaneous inertial frames, as has also often been done, is not equivalent to an accelerating frame. He cites the example for which a charged particle at rest in an inertial frame does not radiate, while a similar particle at rest in an accelerating frame does. As another example, consider again the first of (1.4). If we transform to the inertial frame of a particle with constant acceleration along the x direction,

$$x' = x + \frac{1}{2}at^2,$$

then (1.4) becomes (as in the discussion of rotating frames in [8])

$$ds^2 = (1 - \frac{1}{c^2}a^2t^2)dt^2 - \frac{2}{c^2}atdx'dt - \frac{1}{c^2}(dx'^2 + dy^2).$$

In the frame in which $dx' = dy = dz = 0$, dt is the interval of proper time, and it is not equal to ds. For short times, or small acceleration, the effect is small. We shall discuss this problem further in Section 3.

Continuing for now in the standard framework, Eq. (1.3) effectively eliminates one of the equations of (1.1). We may write

$$\frac{d^2\mathbf{x}}{ds^2} = (\frac{dt}{ds})\frac{d}{dt}(\frac{dt}{ds}\frac{d\mathbf{x}}{dt})$$
$$= \frac{e}{m}\frac{dt}{ds}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}).$$

Cancelling $\gamma = dt/ds$ from both sides, one obtains

$$\frac{d}{dt}(\gamma\mathbf{v}) = \frac{e}{m}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}), \tag{1.6}$$

the starting point for the analysis of Longcope and Sudan [3] and Karamabadi and Angelopoulis [4]. A discrete map can be constructed from (1.6) just as was done for the nonrelativistic equations of Zaslavskii *et el* [1]. As we have remarked above, the stochastic web is found at low energies; it deteriorates at high energies due to the γ factor.

The time component of (1.1) is

$$c\frac{d^2t}{ds^2} = \frac{e}{mc}\mathbf{E} \cdot \mathbf{v}\frac{dt}{ds} \tag{1.7}$$

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or

$$\frac{d\gamma}{dt} = \frac{e}{mc^2}\mathbf{E}\cdot\mathbf{v}. \tag{1.8}$$

Landau and Lifshitz [6] comment that this is a reasonable result, since the "energy" of the particle is γmc^2, and $e\mathbf{E}\cdot\mathbf{v}$ is the work done on the particle by the field. It is important for what we have to say in the following that Eq. (1.7) is not interpretable in terms of the geometry of Lorentz transformations. The second derivative corresponds to an *acceleration* of the observed time variable relative to the "proper time"; the Lorentz transformaton affects only the first derivative, as in (1.4). We understand this equation as an indication that the observed time emerges as a dynamical variable. Mendonça and Oliveira e Silva [8] have studied the relativistic kicked oscillator by introducing a "super Hamiltonian", resulting in a symplectic mechanics of Hamiltonian form, which recognizes that the variables t and E are dynamical variables of the same type as $\mathbf{x}$ and $\mathbf{p}$. This manifestly covariant formulation is equivalent to that of Stueckelberg [9] and Horwitz and Piron [10], which we shall discuss in the next section.

We have computed solutions to the Lorentz force equation for the case of the kicked oscillator (see fig. 1), using methods slightly different from that of Longcope and Sudan [3] and Karimabadi and Angelopoulis [4]. At low velocities, the stochastic web found by Zaslovskii *et al* [1] occurs; the system diffuses in the stochastic region to unbounded energy, as found by Karimabadi and Angelopoulis [4]. The velocity of the particle is light speed limited by the dynamical equations, in particular, by the suppression of the action of the electric field at velocities approaching the velocity of light [5].

The rapid acceleration of the charged particles of the kicked oscillator further suggest that radiation can be an important correction to the motion. The counterexample of Mashoon [11] was based on the phenomenon of radiation. It has been shown by Abraham [12], Dirac [13], Rohrlich [6] and Sokolov and Ternov [14] that the relativistic Lorentz force equations in the presence of radiation reaction is given by the Lorentz-Abraham equation

$$m\ddot{x}^\mu = \frac{e}{c}F^{\mu\nu}\dot{x}_\nu + \frac{2}{3}\frac{r_0}{c}m(\frac{d}{ds}\ddot{x}^\mu - \frac{1}{c^2}\dot{x}^\mu\ddot{x}_\nu\ddot{x}^\nu), \tag{1.9}$$

where $r_0 = e^2/mc^2$, the classical electron radius, and the dots refer here, as in (1.1), to derivatives with respect to s. Note that from the identity (1.3), it follows (by differentiation with respect to s) that

$$\dot{x}_\mu\ddot{x}^\mu = 0, \qquad \dot{x}_\mu\frac{d}{ds}\ddot{x}^\mu + \ddot{x}_\mu\ddot{x}^\mu = 0, \tag{1.10}$$

and hence (1.9) can be written as

$$m\ddot{x}^\mu = \frac{e}{c}F^{\mu\nu}\dot{x}_\nu + \frac{2}{3}\frac{r_0}{c}m\frac{d}{ds}\ddot{x}^\nu(\delta^\mu_\nu + \frac{1}{c^2}\dot{x}^\mu\dot{x}_\nu). \tag{1.11}$$

The last factor on the right is a projection orthogonal to $\dot{x}^\mu$ (if $\dot{x}^\mu\dot{x}_\mu = -c^2$), and therefore (1.11) is consistent with conservation of $\dot{x}^\mu\dot{x}_\mu$. Sokolov and Ternov [4] state that this conservation law follows automatically from (1.9), but it is apparently only consistent.

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Radiation reaction therefore also implies that the connection between proper time and the Lorentz invariant interval may be subject to question.

We have calculated the motion of the kicked oscillator using the form (1.9) of the Lorentz force, corrected for radiaton reaction, undoubtedly a good approximation under certain conditions, and will report on this in another paper in this volume [15].

2. THE STUECKELBERG FORMULATION

As we have remarked above, Mendonça and Oliveira e Silva [8] have used a "super Hamiltonian" formulation to control the covariance of the electromagnetically kicked oscillator. Their formulation of the problem is equivalent to the theory of Stueckelberg [9] and Horwitz and Piron [10]; we shall therefore use the notation of the latter formulation. We first explain the physical basis of this theory, and then derive sytematically the covariant Lorentz force from a model Hamiltonian.

The original thought experiment of Einstein [16] discussed the generation of a sequence of signals in a frame F, according to a clock imbedded in that frame, and their detection by apparatus in a second frame F' in uniform motion with respect to the first. The time of arrival of the signals in F' must be recorded with a clock of the same construction, or there would be no basis for comparison of the intervals between signals sent and those received. It is essential to understand that the clocks in both F and F' run at the same rate. The relation of the interval $\Delta\tau$ between pulses emitted in F and the interval between signals $\Delta\tau'$ received in F', according to the (equivalent) clock in F' is, from the special theory of relativity, given by

$$\Delta\tau' = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.1}$$

This time interval, measured on a "standard" time scale established by these equivalent clocks, is identified to the interval $\Delta t'$, the time interval between signals in the first frame, observed in the second, and called simply the *time* by Einstein. One sees that this Einstein time is subject to distortion due to motion. In general relativity, it is subjuct to distortion due to the gravitational field as well, and in this case the distortion is called the gravitational red-shift. We see that there are essentially *two types of time*, one corresponding to the time intervals at which signals are emitted, and the second, according to the time intervals for which they are detected. The first type, associated with signals that are pre-programmed, is not a dynamical variable, but a given sequence (as for the Newtonian time), and the second, associated with the time at which signals are observed (the Einstein time), is to be understood as a dynamical variable both in classical and quantum theories [17].

Stueckelberg [9] noted that for a free particle, the signals emitted at regular intervals would be recorded at regular intervals in a laboratory, since the free particle would be in motion with respect to the laboratory with the same relation as between F and F'; the motion would then be recorded as a straight line (within the light cone) on a spacetime diagram. In the presence of forces, however, this line could curve. A sufficient deviation from the straight line could make it begin to go backward in time, and then the coordinate t wuld no longer be adequate to parametrize the motion. He therefore

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introduced an invariant parameter τ along the curve, so that there would be a one-to-one corrrespondence between this parameter and the spacetime coordinates. He proposed a Hamiltonian for a free particle of the form (the parameter M provides a dimensional scale, for example, in (2.5); it may also be considered as the Galilean target mass for the *variable* $(1/c)\sqrt{E^2 - c^2\mathbf{p}^2}$)

$$K = \frac{p^\mu p_\mu}{2M} \tag{2.2}$$

for which the Hamilton equations (generalized) give

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} = \frac{p^\mu}{M}. \tag{2.3}$$

It is clear that such a theory is intrinsically "off-shell"; the variables $\mathbf{p}$ and $p^0 = E/c$ are independent, as are the observables $\mathbf{x}$ and t, so that the phase space is eight-dimensional. Dividing the equation for the space indices by the equation for the time index, one obtains

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = c^2 \frac{\mathbf{p}}{E}, \tag{2.4}$$

precisely the Einstein formula for velocity. Furthermore, for the time component,

$$\frac{dt}{d\tau} = \frac{E}{Mc^2}; \tag{2.5}$$

in case the particle is "on-shell", so that $Mc^2 = \sqrt{E^2 - c^2\mathbf{p}^2}$, (2.5) reads

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}},$$

coinciding with (2.1).

Stueckelberg [9] then considered adding a potential term $V(x)$, to treat one-body mechanics, and the gauge substitution $p^\mu \to p^\mu - eA^\mu(x)$ for the treatment of problems with electromagnetic interaction. He proposed a quantum theory, for which the Hamiltonian generates a Schrödinger type evolution

$$i\hbar \frac{\partial}{\partial \tau} \psi_\tau(x) = K \psi_\tau(x). \tag{2.6}$$

Horwitz and Piron [10] generalized the Stueckelberg theory for application to many body problems. They assumed that the standard clocks constitute a universal time, as for the Robertson-Walker time (the Hubble time) of general relativity [18], so that separate subsystems are correlated in this time. In this framework, it became possible to solve, for example, the two body problem in both classical [10] and quantum theory [19].

The equations (1.1) are not generally derived rigorously from a well-defined Lagrangian or Hamiltonian. They result from a relativistic generalization of the nonrelativistic Lorentz force (which is derivable from a nonrelativistic Hamiltonian). In the following, we shall

derive these equations rigorously from the Stueckelberg theory, to emphasize more strongly the nature of the problem we have discussed above, and to clarify some important points.

The Hamiltonianian form for a particle with electromagnetic interaction proposed by Stueckelberg [9] is

$$K = \frac{(p^\mu - \frac{e}{c}A^\mu(x))(p_\mu - \frac{e}{c}A_\mu(x))}{2M}.$$

(2.7)

The equation of motion for x^μ is (we use the upper dot from now on to denote differentiation by τ, the universal invariant time)

$$\dot{x}^\mu = \frac{\partial K}{\partial p_\mu} = \frac{(p^\mu - \frac{e}{c}A^\mu(x))}{M}$$

(2.8)

and we see that then

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = -c^2\left(\frac{ds}{d\tau}\right)^2 = \frac{(p^\mu - \frac{e}{c}A^\mu(x))(p_\mu - \frac{e}{c}A_\mu(x))}{M},$$

(2.9)

a quantity proportional to K, and therefore strictly conserved. In fact, this quantity is the gauge invariant mass-squared:

$$(p^\mu - \frac{e}{c}A^\mu(x))(p_\mu - \frac{e}{c}A_\mu(x)) = -m^2c^2,$$

(2.10)

where we define m as the dynamical mass, a constant of the motion. It then follows that

$$c^2\left(\frac{ds}{d\tau}\right)^2 = c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{d\mathbf{x}}{d\tau}\right)^2 = \frac{m^2c^2}{M^2}$$

(2.11)

and, extracting a factor of $(dt/d\tau)$,

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{m^2/M^2}{1 - \frac{v^2}{c^2}}.$$

(2.12)

Up to a constant factor, the Stueckelberg theory therefore maintains the identity (1.3).

We now derive the Lorentz force from the Hamilton equation (this derivation has also been carried out by C. Piron [20]). The Hamilton equations for energy momentum are

$$\begin{aligned}\frac{dp^\mu}{d\tau} &= -\frac{\partial K}{\partial x_\mu} = \frac{(p^\nu - \frac{e}{c}A^\nu)}{M}\left(\frac{e}{c}\frac{\partial A_\nu}{\partial x_\mu}\right) \\ &= \frac{e}{c}\frac{dx^\nu}{d\tau}\frac{\partial A_\nu}{\partial x_\mu}.\end{aligned}$$

(2.13)

Since $p^\mu = M\frac{dx^\mu}{d\tau} + \frac{e}{c}A^\mu$, the left hand side is (A^μ is evaluated on the particle world line $x^\nu(\tau)$)

$$\frac{dp^\mu}{d\tau} = M\frac{d^2x^\mu}{d\tau^2} + \frac{e}{c}\frac{\partial A^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau},$$

(2.14)

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and hence

$$M\frac{d^2x^\mu}{d\tau^2} = \frac{e}{c}\Big(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x^\nu}\Big)\frac{dx^\nu}{d\tau},$$

or

$$M\frac{d^2x^\mu}{d\tau^2} = \frac{e}{c}F^\mu_\nu\frac{dx^\nu}{d\tau}, \tag{2.15}$$

where $(\partial^\mu \equiv \partial/\partial x_\mu)$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2.16}$$

The form of (2.15) is identical to that of (1.4), but the temporal derivative is not with respect to the variable s, the Minkowski distance along the particle trajectory, but with respect to the universal evolution parameter τ.

One might argue that these should be equal, or at least proportional by a constant, since the proper time is equal to the time which may be read on a clock on the particle in its rest frame. For an accelerating particle, however, *one cannot transform by a Lorentz transformation, other than instantaneously, to the particle rest frame.* It appears, therefore, that the formula (2.15) could have a more reliable interpretation. The parameter of evolution τ does not require a Lorentz transformation to achieve its meaning.

Since m^2 is absolutely conserved by the Hamiltonian model (2.7), however, we have the constant relation

$$ds = \frac{m}{M}d\tau, \tag{2.17}$$

assuming the positive root (as we shall also do for the root of (2.12); we do not wish to discuss the antiparticle solutions here). Eq. (2.15) can therefore be written exactly as (1.1).

We see that the Stueckelberg formulation in terms of an absolute time does not avoid the serious problem of consistency that we have pointed out before. It is clear that the difficulty is associated with the fact that the Stueckelberg Hamiltonian, as we have written it, preserves the mass-shell, and we therefore understand the identity (1.3) as a *mass-shell* relation.

Returning to the Stueckelberg-Schrödinger equation (2.6), we see that the gauge invariant replacement $p^\mu \to p^\mu - \frac{e}{c}A^\mu(x)$ is not adequate. The additional derivative on the left hand side of (2.6) must also be replaced by a gauge covariant term. The possibility of τ dependence in the gauge transformation implies that the gauge fields themselves may depend on τ. The gauge covariant equation should then be [21]

$$i\frac{\partial}{\partial\tau}\psi_\tau(x) = \Big\{\frac{1}{2M}(p_\mu - \frac{e_0}{c}a_\mu)(p^\mu - \frac{e_0}{c}a^\mu) - \frac{e_0}{c}a_5\Big\}\psi_\tau(x), \tag{2.18}$$

, where the fields a_α, $\alpha = (0, 1, 2, 3, 5)$ with $\partial_5 \equiv \partial/\partial\tau$, change under the gauge transformation $\psi \to \exp i\frac{e_0}{c}\Lambda\psi$ according to $a_\alpha \to a_\alpha - \partial_\alpha\Lambda$. It follows from this equation, in a way analogous to the Schrödinger non-relativistic theory, that there is a current

$$j^\mu_\tau = -\frac{i}{2M}\{\psi^*_\tau(\partial^\mu - i\frac{e_0}{c}a^\mu)\psi_\tau - \psi_\tau(\partial^\mu + i\frac{e_0}{c}a^\mu)\psi^*_\tau\}, \tag{2.19}$$

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which, with

$$\rho_\tau \equiv j_\tau^5 = |\psi_\tau(x)|^2,$$

satisfies

$$\partial_\tau \rho_\tau + \partial_\mu j_\tau^\mu \equiv \partial_\alpha j^\alpha = 0. \tag{2.20}$$

We see that for $\rho_\tau \to 0$ pointwise ($\int \rho_\tau(x)d^4x = 1$ for any τ),

$$J^\mu(x) = \int_{-\infty}^{\infty} j_\tau^\mu(x)d\tau \tag{2.21}$$

satisfies

$$\partial_\mu J^\mu(x) = 0, \tag{2.22}$$

and can be a source for the standard Maxwell fields. Since the field equations are linear, with source j^α, one identifies the integral $\int d\tau a^\mu(x,\tau)$ (or, alternatively, the 0-mode) with the Maxwell potentials [21]. It then follows that the so-called pre-Maxwell fields a^α have dimension L^{-2}, and that the charge e_0 has dimension L. The Lagrangian density for the fields, quadratic in the field strengths ($\alpha, \beta = 0, 1, 2, 3, 5$)

$$f^{\alpha\beta} = \partial^\alpha a^\beta - \partial^\beta a^\alpha,$$

which has dimension L^{-3}, must carry a dimensional parameter, say λ, and from the field equations $\lambda \partial_\alpha f_\alpha^\beta = e_0 j^\beta$, one sees that the Maxwell charge is $e = e_0/\lambda$ [21].

We understand the operator on the right hand side of (2.18) as the quantum form of a classical evolution function

$$K = \frac{1}{2M}(p_\mu - \frac{e_0}{c}a_\mu)(p^\mu - \frac{e_0}{c}a^\mu) - \frac{e_0}{c}a_5. \tag{2.23}$$

It follows from the Hamilton equations that

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu - \frac{e_0}{c}a^\mu}{M} \tag{2.24}$$

and

$$\frac{dp^\mu}{d\tau} = \frac{e_0}{c}\frac{dx^\nu}{d\tau}\frac{\partial a_\nu}{\partial x_\mu} + \frac{e_0}{c}\frac{\partial a_5}{\partial x_\mu}.$$

Hence,

$$M\frac{d^2x^\mu}{d\tau^2} = \frac{e_0}{c}\frac{dx^\nu}{d\tau}f_\nu^\mu + \frac{e_0}{c}\Big(\frac{\partial a_5}{\partial x_\mu} - \frac{\partial a^\mu}{\partial \tau}\Big). \tag{2.25}$$

If we define $x^5 \equiv \tau$, the last term can be written as $\partial^\mu a_5 - \partial_5 a^\mu = f_5^\mu$, so that

$$M\frac{d^2x^\mu}{d\tau^2} = \frac{e_0}{c}\frac{dx^\nu}{d\tau}f_\nu{}^\mu + \frac{e_0}{c}f_5^\mu. \tag{2.26}$$

Note that in this equation, the last term appears in the place of the radiation correction terms of (1.9). It plays the role of a generalized electric field. Furthermore, we see that

the relation (1.3), consistent with the standard Maxwell theory, no longer holds as an identity; the Stueckelberg form of this result (2.11) in the presence of standard Maxwell fields, where m^2 is conserved, is also not generally valid. We now have

$$\frac{d}{d\tau}\frac{1}{2}M\left(\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}\right) = \frac{e_0}{c}\frac{dx^\mu}{d\tau}f_{\mu 5},$$
(2.27)

and does not vanish. The right hand side corresponds to mass transfer from the field to the particle.

As for the method of Longcope and Sudan [3], we may transform the derivatives with respect to τ to derivatives with respect to t in the equation (2.26) as follows. Defining $\zeta = dt/d\tau$, it follows from (2.26) that there is an additional term in the analogous form of the rate of change of ζ (we use lower case to denote the pre-Maxwell field strengths),

$$\frac{d\zeta}{dt} = \frac{e_0}{Mc^2}(\mathbf{e}\cdot\mathbf{v}) + \frac{e_0}{\zeta Mc^2}f^0{}_5$$
(2.28)

The space components of (2.26) can be written as

$$\frac{d^2x^j}{dt^2} = \frac{e_0}{\zeta M}\left[e^j + \frac{1}{c}(\mathbf{v}\times\mathbf{h})^j - \frac{v^j}{c^2}(\mathbf{e}\cdot\mathbf{v})\right]$$
$$+ \frac{e_0}{Mc\zeta^2}\left[f^j{}_5 - \frac{v_j}{c}f^0{}_5\right].$$
(2.29)

To illustrate some of the properties of this system of equations, we treat a simple example in Appendix A. The effective additonal forces include not only the term associated with the work done by the field, but additonal terms associated specifically with the τ-dependence of the fields, and the fifth (scalar) field a_5. Given the fields $f^\alpha{}_\beta$, Eqs.(2.28) and (2.29) form a nonlinear coupled system of equations for the particle motion.

For a gauge (generalized Lorentz) in which $\partial_\alpha a^\alpha = 0$, the field equations [21]

$$\partial_\alpha f^{\beta\alpha} = ej^\beta$$

become

$$-\partial_\alpha\partial^\alpha a^\beta = ej^\beta,$$

where, classically, $j^\beta = \dot{x}^\mu\delta^4(x-x(\tau))$, $\rho = \delta^4(x-x(\tau))$, and $x \equiv x^\mu(\tau)$ is the world line. The analysis of these equations is in progress.

It has recently been shown that, with the help of the Green's functions for the wave equations of the fields in x^μ, τ, that the self-reaction derived from the contributions on the right hand side of (2.26) are precisely of the form of the radiation reaction terms in the Abraham-Lorentz equations (1.9) in the limit that the theory is constrained to mass shell, *i.e.*, that (1.3) is enforced [22]. The off-shell corrections provided by (2.26) make the system of equations consistent, and should therefore provide a basis for computing problems involving the interaction of radiation with relativistic particles in a consistent way.

CONCLUSIONS

We have shown that the standard relativistic Lorentz force equations are not consistent since they imply the mass-shell constraint $\dot{x}^\mu \dot{x}_\mu = -c^2$, a relation that can be valid only for a charged particle moving at constant velocity. The corrections are generally small for short times or small accelerations, and therefore calculations made with this Lorentz force are, in many applications, quite acceptable. However, for very large accelerations (*at* large compared to c), they could become inaccurate.

A consistent theory may be constructed from a fully gauge covariant form of the Stueckelberg [9][10] manifestly covariant dynamics, a theory which introduces a fifth gauge field [21]. The Lorentz invariant force equation derived from this theory contains an additional term which enters in a way analogous to the radiation reaction term in the Abraham-Lorentz-Dirac equation (the self reaction force derived from this generalized equation in the mass-shell limit coincides with the radiation reaction term obtained by quite different methods for the Abraham-Lorentz-Dirac equation; it contains contributions from both terms on the right hand side [22]).

It appears that the consistency of the *classical* equations governing the interaction of charged particles with electromagnetic radiation requires that both the particles and the fields must be permitted to move "off-shell", as in the vertices of quantum field theory.

Acknowledgements

Appendix A

The purpose of the following example is to show that in some cases the fifth field $f^\mu{}_5$ can cause to an effect which is very similar to the radiation effect that is calculated by Lorentz-Dirac equation. The fact that the mass is not conserved (the off-mass-shell case) is equivalent, in the case of radiation, to loss of energy through the radiation process. The particular example that we treat here is that of a charged particle in the presence of an uniform magnetic field in z direction ($\mathbf{B} = (0, 0, B_0)$).

As for the radiation reaction term of the Lorentz-Dirac equation, we choose the fifth field term to be *

$$f^\mu{}_5 = (C_1 \dot{t}, C_2 \dot{x}, C_2 \dot{y}, 0), \tag{A.1}$$

where the dot indicates derivative with respect to τ. The Lorentz force (2.26) can be written as a set of differential equations,

$$M \frac{d^2 t}{d\tau^2} = \frac{e}{c} C_1 \frac{dt}{d\tau} \tag{A.2}$$

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* It appears that for the usual form of the radiation reaction, in an example with the field magnitudes that we shall choose, the $\frac{d\dddot{x}^\mu}{d\tau}$ term seems to be negligible, and the $\ddot{x}^\mu \ddot{x}_\mu$ may be approximated by a constant number; one is left with the $\dot{x}^\mu$ term. We choose the fifth field term to have a similar structure. This choice is appropriate due to the close relation of these the radiation reaction terms of the usual theory [22].

$$M\frac{d^2x}{d\tau^2} = \frac{eB_0}{c}\frac{dy}{d\tau} + \frac{e}{c}C_2\frac{dx}{d\tau} \qquad (A.3)$$

$$M\frac{d^2y}{d\tau^2} = -\frac{eB_0}{c}\frac{dx}{d\tau} + \frac{e}{c}C_2\frac{dy}{d\tau}. \qquad (A.4)$$

The solution of Eq. $(A.2)$ is

$$\dot{t} = \dot{t}_0 e^{\alpha_1\tau}, \qquad (A.5)$$

where $\alpha_1 = \frac{eC_1}{Mc}$. Using the complex coordinate $u = \dot{x} + i\dot{y}$, Eqs. $(A.3)$ and $(A.4)$ can be written as

$$\frac{du}{d\tau} = -i\Omega u + \alpha_2 u, \qquad (A.6)$$

where $\alpha_2 = \frac{eC_2}{Mc}$ and $\Omega = \frac{eB_0}{Mc}$ (the Larmor frequency). The solution is

$$u = u_0 \exp^{\alpha_2\tau} e^{-i\Omega\tau}. \qquad (A.7)$$

Using $u(\tau)$ one finds that,

$$\dot{x} = e^{\alpha_2\tau}(\dot{x}_0\cos(\Omega\tau) + \dot{y}_0\sin(\Omega\tau))$$
$$\dot{y} = e^{\alpha_2\tau}(-\dot{x}_0\sin(\Omega\tau) + \dot{y}_0\cos(\Omega\tau)). \qquad (A.8)$$

As expected, the radiation effect is determined by the constants α_1 and α_2.

It is possible to calculate the actual velocities by dividing Eqs. $(A.8)$ by $\dot{t}$; this results in

$$\frac{dx}{dt} = e^{-\alpha\tau}\left(\left(\frac{dx}{dt}\right)_0\cos(\Omega\tau) + \left(\frac{dy}{dt}\right)_0\sin(\Omega\tau)\right)$$
$$\frac{dy}{dt} = e^{-\alpha\tau}\left(-\left(\frac{dx}{dt}\right)_0\sin(\Omega\tau) + \left(\frac{dy}{dt}\right)_0\cos(\Omega\tau)\right), \qquad (A.9)$$

where $\alpha = \alpha_1 - \alpha_2$. Notice that when $\alpha_1 = \alpha_2$, there is apparent radiation (decrease of amplitude) as a function of τ but not as a function of t ; in terms of t (which is redshifted) the particle appears to be circling forever on the same circle. This remarkable illustration is somewhat analogous to the phenomenon in which there is an infinite time required for a particle to arrive at the Schwarzschild radius in the Schwarzschild coordinate t, but a finite interval in the proper time of the particle.

The magnitude of the (t-) velocity of the particle is

$$v = v_0 e^{-\alpha\tau}. \qquad (A.10)$$

When $\alpha = \frac{1}{\tau_0}$, where $\tau_0 = \frac{1}{\gamma_0\Omega^2}$ (γ_0 is the radiation constant of the Lorentz-Dirac equation), Eq. $(A.10)$ is exactly the solution which was obtained using the Lorentz-Dirac equation [14][15]. This result is consistent with the approximations we have made in constructing the example.

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Figure Caption

fig. 1. A typical relativstic stochastic web.

Fig. 1

V_x

V_y